THETA DIVISORS OF STABLE VECTOR BUNDLES MAY BE NONREDUCED

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ABSTRACT. A generic strictly semistable bundle of degree zero over a curve X has a reducible theta divisor, given by the sum of the theta divisors of the stable summands of the associated graded bundle. The converse is not true: Beauville and Raynaud have constructed stable bundles with reducible theta divisors. For X of genus $g \geq 5$, we construct stable vector bundles of all ranks $r \geq 5$ over X with reducible and nonreduced theta divisors. We also adapt the construction to symplectic bundles.

1. INTRODUCTION

Let X be a complex projective smooth curve of genus $g \geq 2$. We write J for the Jacobian variety parametrizing line bundles of degree $g-1$ over X. To a vector bundle $V \to X$ of degree zero we may associate the set

$$\{N \in J : h^0(N \otimes V) > 0\}. \tag{1.1}$$

If V is a generic semistable bundle of rank n, then this is the support of a divisor Θ_V on J, called the *theta divisor of* V, which is algebraically equivalent to $n\Theta$. See Beauville [2, 3] for details[1]. Laszlo has given an analogue of the Riemann singularity theorem for Θ_V:

Theorem 1.1. *If* Θ_V *is defined, then* $\mathrm{mult}_N \Theta_V \geq h^0(N \otimes V)$.

Proof. On the moduli space $\mathcal{U}(n, n(g-1))$ of semistable bundles of rank n and degree $n(g-1)$, there is a canonically defined theta divisor Δ, generalizing $\Theta \subset J$, whose support consists of bundles with nonzero sections. The divisor Θ_V is the pullback of Δ via the map $J \to \mathcal{U}(n, n(g-1))$ defined by $N \mapsto N \otimes V$, when this is a divisor. Laszlo shows in [11, Corollaire II.2] that

$$\mathrm{mult}_{N \otimes V} \Delta \geq h^0(N \otimes V).$$

The statement then follows by functoriality. □

If V is strictly semistable, S-equivalent to a decomposable bundle $\bigoplus_i V_i$ where each V_i is stable of degree zero, then Θ_V has the reducible theta divisor $\sum_i \Theta_{V_i}$ (when this exists). The converse is not true: Raynaud gave an example of a stable rank two bundle over a special curve of genus 3 with a reducible theta divisor (details in [12]). In [3], Beauville constructed stable bundles of higher rank with reducible theta divisors over a general curve of genus $g \geq 3$.

[1]For certain nongeneric semistable or stable V, then (1.1) is the whole of J. This phenomenon was first studied by Raynaud [13], and has subsequently attracted a good deal of attention.

By Theorem 1.1, the polystable bundle $V^{\oplus n}$ has nonreduced theta divisor $n\Theta_V$. In light of Beauville's and Raynaud's constructions, it is reasonable to expect that there also exist stable bundles with nonreduced theta divisors. In §3, we construct examples of such bundles, with arbitrarily large rank ≥ 5. Precisely, we show:

Theorem 1.2. *Suppose X has genus $g \geq 5$. Let Θ_V be the theta divisor of a generic stable bundle V of rank $n \geq 1$ over X. Let t be a positive integer with $t < n(g-1)$. Then for any rank $r \geq tn + 2$, there exist stable bundles W of degree zero and rank r such that Θ_W exists and contains $(t-1)\Theta_V$ as a subscheme.*

In particular, letting $t = 2$ or 3 and $n = 1$, we obtain:

Corollary 1.3. *Over any curve X of genus $g \geq 5$, there exist stable bundles W of all ranks $r \geq 4$ (resp., $r \geq 5$) with reducible (resp., reducible and nonreduced) theta divisor.*

These W are obtained as extensions $0 \to E \to W \to M \to 0$ where E is a stable bundle of degree -1 with "large" families of maximal subbundles for certain ranks. These E, which we construct in §2, are similar to examples of Ballico and Russo [1] of bundles whose Quot scheme $M_k(E)$ of maximal subbundles of rank k is of large dimension.

In §5, we adapt the construction to produce symplectic bundles W of even rank ≥ 6 with reducible theta divisors, and nonreduced if $r \geq 8$. These are obtained as extensions $0 \to E \to W \to E^* \to 0$ where E is as above. As background, in §4 we obtain some results on liftings in symplectic extensions which we hope may also be applicable in other contexts.

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2. Bundles with many maximal subbundles

In this section, we construct the bundles E referred to in the introduction. We begin by recalling some results on vector bundle extensions.

2.1. **Extensions, lifting and geometry.**

Let E and F be vector bundles over a curve, and let $0 \to E \to W \to F \to 0$ be a nontrivial extension. In this section we recall some results on liftings of elementary transformations of F to W.

Let V be a vector bundle with $h^1(V) \neq 0$, and write π for the projection $\mathbb{P}V \to X$. By Serre duality and the projection formula and since $\pi_* O_{\mathbb{P}V}(1) = V^*$, we have an identification

$$H^1(X, V) \xrightarrow{\sim} H^0(\mathbb{P}V, \pi^* K_X \otimes O_{\mathbb{P}V}(1))^*.$$

By standard algebraic geometry, we obtain a map $\mathbb{P}V \dashrightarrow \mathbb{P}H^1(V)$. See [6, §2] for more information and other descriptions of this map.

If $V = \mathrm{Hom}(F, E) = F^* \otimes E$ then we may consider the locus $\Delta_{F^* \otimes E}$ of rank one tensors, which has dimension $\operatorname{rk} F + \operatorname{rk} E - 1$.

Criterion 2.1. *Let E, F and W be as above. If an elementary transformation*

$$0 \to \tilde{F} \to F \to \tau \to 0$$

with $\deg \tau \leq k$ lifts to a subsheaf of W, then the class $\delta(W)$ of the extension belongs to $\mathrm{Sec}^k \psi(\Delta_{F^* \otimes E})$.

Proof. This is proven in [5, Theorem 4.4 (i)]. Note that in [5] there are various assumptions on the degrees and genericity of E and F, which need not be satisfied in the present applications. However, the function of these assumptions is to ensure that $\mathbb{P}\mathrm{Hom}(F, E)$ is *embedded* in $\mathbb{P}H^1(\mathrm{Hom}(F, E))$, which is not required here. □

2.2. **Construction of bundles with many maximal subbundles.** Let X be a curve of genus $g \geq 5$. Here we construct the bundles E mentioned in the introduction. Like the bundles with large $M_k(E)$ constructed by Ballico and Russo [1], these E will be extensions of a decomposable bundle by a line bundle.

Choose a generic stable bundle $V \to X$ of degree zero and rank $n \geq 1$, and a positive integer t with

$$t < n(g-1). \tag{2.1}$$

Let $L \to X$ be a line bundle of degree -1, and consider a generic extension $0 \to L \to E \to V \otimes \mathbb{C}^t \to 0$. The following two lemmas form a partial analogue of the Claim in the proof of [1, Theorem 0.0.1]:

Lemma 2.2. *Every subbundle of E has negative degree.*

Proof. Let F be a proper subbundle of E. Then F fits into a diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & E \arrow[r] & V \otimes \mathbb{C}^t \arrow[r] & 0 \\ 0 \arrow[r] & F_1 \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

where F_1 is either zero or L, and F_2 is a subsheaf of $V \otimes \mathbb{C}^t$. If $F_1 = L$ or if $t = 1$ then clearly F has negative degree. If $t \geq 2$, we must show that no subbundle of the form $V \otimes \Lambda$ lifts to E, where $\Lambda \subseteq \mathbb{C}^t$ is a proper vector subspace. Clearly it suffices to treat the case $\dim \Lambda = 1$. Thus we need to check that subspaces of the form

$$\mathrm{Ker}\left(H^1(\mathrm{Hom}(V \otimes \mathbb{C}^t, L)) \to H^1(\mathrm{Hom}(V \otimes \Lambda, L))\right) \tag{2.2}$$

do not sweep out $H^1(\mathrm{Hom}(V \otimes \mathbb{C}^t, L))$. Since V is stable, $h^0(\mathrm{Hom}(V, L)) = 0$. The dimension of (2.2) is therefore $(t-1)h^1(\mathrm{Hom}(V, L))$. Furthermore, the subspaces Λ vary in $\mathbb{P}^{t-1}$. Hence it suffices to check that

$$(t-1) + (t-1)h^1(\mathrm{Hom}(V, L)) < t \cdot h^1(\mathrm{Hom}(V, L)),$$

that is, $t - 1 < h^1(\mathrm{Hom}(V, L))$.

By Riemann–Roch and since $h^0(\mathrm{Hom}(V, L)) = 0$, we have $h^1(\mathrm{Hom}(V, L)) = ng$. The inequality $t - 1 < ng$ follows from assumption (2.1), and we are done. □

Lemma 2.3. *Let E be a generic extension of $V \otimes \mathbb{C}^t$ by L as above. Then all degree -1 subbundles of E contain the subbundle L.*

Proof. We proceed by induction on t. It is convenient to begin with the case $t = 1$, although in applications we will most often assume that $t \geq 2$.

Consider an extension $0 \to L \to E \to V \to 0$. Any degree -1 subbundle F of E not containing L must lift from a subsheaf of V.

Proposition 2.4. *Suppose X has genus $g \geq 5$. Then a generic vector bundle V of rank $n \geq 2$ and degree zero has no subbundles of degree -1.*

Proof. By Russo–Teixidor i Bigas [14, Theorem 0.2], the Quot scheme of subsheaves of degree -1 and rank m of a generic V is empty when the expected dimension $n - m(n-m)(g-1)$ is negative. One checks easily that the maximum value of this dimension occurs at $m = 1$ and $m = n-1$, when it is equal to $n - (n-1)(g-1)$. Since $g \geq 5$, this would follow from $n - 4(n-1) < 0$, which is clear since $n \geq 2$. $\square$

By the proposition, any subbundle $F \subset E$ of degree -1 not containing L must lift from an elementary transformation $F \to V \to \mathbb{C}_x$. By Theorem 2.1, this happens only if the extension class of E belongs to the image of the scroll $\mathbb{P}\mathrm{Hom}(V, L)$ in $\mathbb{P}H^1(\mathrm{Hom}(V, L))$ (note that since L is a line bundle, $\Delta_{V^* \otimes L} \cong \mathbb{P}\mathrm{Hom}(V, L)$). This scroll has dimension at most n, whereas $h^1(\mathrm{Hom}(V, L)) - 1 = ng - 1$. Since $g \geq 5$, a general extension class $\delta(E)$ does not belong to the scroll. Thus we have proven the lemma for $t = 1$.

Now suppose $t \geq 2$, and let E be a generic extension $0 \to L \to E \to V \otimes \mathbb{C}^t \to 0$. Choose a subspace $\Lambda \subset \mathbb{C}^t$ of dimension $t-1$, and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & V \otimes \Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & V \otimes \mathbb{C}^t \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & V \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since the induced map $H^1(\mathrm{Hom}(V \otimes \mathbb{C}^t, L)) \to H^1(\mathrm{Hom}(V \otimes \Lambda, L))$ is surjective, we may assume that E_0 is a generic extension of $V \otimes \Lambda$ by L.

Suppose $F \subset E$ is a proper subbundle of degree -1. Then we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & E_0 \arrow[r] & E \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where F_1 is a subbundle of E_0 and F_2 a subsheaf of V.

Firstly, suppose $F_1 \neq 0$. By Lemma 2.2 we have $\deg F_1 \leq -1$, and therefore $\deg F_2 \geq 0$. Since V is stable, the only possibilities are $F_2 = 0$ and $F_2 = V$, and so in fact $\deg F_1 = -1$. By induction, L belongs to F_1 and hence to F.

On the other hand, if $F_1 = 0$ then $F \cong F_2$ must lift from a degree -1 subsheaf of V. By Proposition 2.4, the only possibility is that F_2 is an elementary transformation

$$0 \to F \to V \to \tau \to 0 \tag{2.3}$$

where τ is a torsion sheaf of degree 1. By Theorem 2.1, the lifting of such an F implies that the class ε of the extension

$$0 \to E_0 \to E \to V \to 0$$

belongs to the image of the scroll $\mathbb{P}\mathrm{Hom}(V, E_0)$ in $\mathbb{P}H^1(\mathrm{Hom}(V, E_0))$.

We note that since E/L is a direct sum $V \otimes \mathbb{C}^t$, the class ε belongs to

$$\mathrm{Ker}\left(H^1(\mathrm{Hom}(V, E_0)) \to H^1(\mathrm{Hom}(V, V \otimes \Lambda))\right),$$

which has dimension

$$h^1(\mathrm{Hom}(V, L)) - h^0(\mathrm{Hom}(V, V \otimes \Lambda)) = ng - (t-1).$$

Conversely, it is easy to see that any element of this kernel gives an extension E of the form we began with.

We claim now that the intersection of $\psi\left(\Delta_{V^* \otimes E_0}\right)$ with

$$\mathrm{Im}\left(\mathbb{P}H^1(\mathrm{Hom}(V, L)) \dashrightarrow \mathbb{P}H^1(\mathrm{Hom}(V, E_0))\right)$$

is exactly $\psi\left(\mathbb{P}\mathrm{Hom}(V, L)\right)$. Suppose $\psi(v^* \otimes e_0) \in \mathbb{P}H^1(\mathrm{Hom}(V, L))$ where $e_0 \notin L$. Write v_0 for the image of e_0 in $V \otimes \Lambda$. Then the corresponding point $v^* \otimes v_0$ is a base point of the natural map

$$\mathbb{P}(\mathrm{End}(V) \otimes \Lambda) \dashrightarrow \mathbb{P}H^1(\mathrm{End}(V) \otimes \Lambda).$$

But it follows from Hwang–Ramanan [9, Proposition 3.2] that this map is base point free (in fact an embedding) for general V. Thus if $\psi(v^* \otimes e_0) \in \mathbb{P}H^1(\mathrm{Hom}(V, L))$ then in fact $e_0 \in L$.

By the claim, we must check that

$$\dim \mathbb{P}\mathrm{Hom}(V, L) < h^1(\mathrm{Hom}(V, L)) - h^0(\mathrm{Hom}(V, V \otimes \Lambda)) - 1,$$

that is, $n < ng - (t-1) - 1$. This follows from assumption (2.1). Hence a generic extension $0 \to E_0 \to E \to V \to 0$ of our preferred type admits no lifting of the form (2.3), and we are done. ☐

Corollary 2.5. (1) *Any subbundle $F \subseteq E$ of degree -1 is of the form $0 \to L \to F \to V \otimes \Lambda \to 0$, where Λ is a uniquely determined vector subspace of $\mathbb{C}^t$.*

(2) *The degree -1 subbundles of E are parametrised by the union of the Grassmann varieties $\mathrm{Gr}(\mathbb{C}^t, s)$ for $s \in \{0, \ldots, t\}$.*

Proof. This is straightforward to check, in view of Lemma 2.3 and since V is stable. ☐

3. Stable bundles with reducible and nonreduced theta divisors

Proposition 3.1. *Let M be a generic stable bundle of rank $m \geq 1$ and degree 1. Then any proper subbundle of M has negative degree.*

Proof. Similar to Proposition 2.4. □

Theorem 3.2. *A generic extension $0 \to E \to W \to M \to 0$ is a stable vector bundle.*

Proof. Suppose $G \subset W$ is a proper subbundle. If G is contained in the subbundle E then $\deg G \leq -1$ by Lemma 2.2. Otherwise, we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] & W \arrow[r] & M \arrow[r] & 0 \\ 0 \arrow[r] & F \arrow[r] \arrow[u] & G \arrow[r] \arrow[u] & H \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

where F is a subbundle of E and H a subsheaf of M. By Proposition 3.1, it suffices to exclude liftings of the following types to W:

(i) extensions $0 \to F \to G \to M \to 0$ where $F \subset E$ has degree -1; and
(ii) degree zero elementary transformations of M.

(i) Suppose $F \subset E$ is a proper subbundle of degree -1. By Corollary 2.5 (1), we have $\mathrm{rk}\, F = sn + 1$ for some $0 \leq s < t$. Then an extension G of M by F belongs to W if and only if $\delta(W)$ belongs to

$$\mathrm{Im}\left(H^1(\mathrm{Hom}(M, F)) \to H^1(\mathrm{Hom}(M, E))\right).$$

By Corollary 2.5 (2), to exclude case (i) in general, it will suffice to show that

$$h^1(\mathrm{Hom}(M, E)) - h^1(\mathrm{Hom}(M, F)) - \dim \mathrm{Gr}(\mathbb{C}^t, s) > 0.$$

A straightforward calculation using Riemann–Roch shows that this would follow from $((m(g-1)+1)n - s)(t-s) > 0$. We may assume $t > s$, since W is a nontrivial extension. Then $s < t < n(g-1)$ by (2.1), and then clearly $((m(g-1)+1)n-s)(t-s) > 0$ as desired.

(ii) By Theorem 2.1, a degree 0 elementary transformation of M lifts to W only if the extension class of W belongs to $\psi\left(\Delta_{M^* \otimes E}\right)$ in $\mathbb{P}H^1(\mathrm{Hom}(M, E))$. We have

$$\begin{aligned} \dim \psi\left(\Delta_{M^*\otimes E}\right) &\leq m + tn + 1 \\ &< m + tn + 1 + m(tn+1)(g-1) - 1 = \dim \mathbb{P}H^1(\mathrm{Hom}(M, E)), \end{aligned}$$

whence a general extension W admits no such lifting. □

Now we study theta divisors of such extensions W. Suppose $t \geq 2$, and let V, L, E and M be as above. Since V is generic, we may assume V has a reduced theta divisor Θ_V.

Theorem 3.3. *A generic extension $0 \to E \to W \to M \to 0$ has a reducible theta divisor Θ_W which scheme-theoretically contains $(t-1)\Theta_V$. In particular, if $t \geq 3$ then Θ_W is also nonreduced.*

Proof. For $P \in \Theta_V$, consider the exact sequence

$$0 \to P \otimes L \to P \otimes E \to (P \otimes V) \otimes \mathbb{C}^t \to 0.$$

Since L and V are generic, we may assume for generic $P \in \Theta_V$ that $h^0(P \otimes L) = 0$. As Θ_V is reduced, furthermore $h^0(P \otimes V) = 1$ for generic $P \in \Theta_V$ by Theorem 1.1. Taking global sections, we obtain

$$0 \to H^0(P \otimes E) \to H^0(P \otimes V) \otimes \mathbb{C}^t \to H^1(P \otimes L) \to \cdots$$

By Riemann–Roch, $h^1(P \otimes L) = 1$. By exactness, $h^0(P \otimes E) \geq t - 1$ for all $P \in \Theta_V$. Since $E \subset W$, we have $h^0(P \otimes W) \geq t - 1$ for all $P \in \Theta_V$. Thus by Theorem 1.1, the theta divisor Θ_W, if defined, has multiplicity at least $t - 1$ at all $P \in \Theta_V$, and thus is of the form $(t-1)\Theta_V + R_W$ where R_W is a divisor on J algebraically equivalent to $(n + 1 + m)\Theta$.

Now we check that a generic such W has a theta divisor which is reducible. Since L is generic of degree -1, we have $h^0(N \otimes L) = 0$ for all $N \in J$ outside a locus of codimension at least 2. Therefore, for generic $N \in J \backslash \operatorname{Supp} \Theta_V$, any map $N^{-1} \to W$ must lift from a map $N^{-1} \to M$. By Hirschowitz's Lemma [14, Theorem 1.2] and Riemann–Roch, we have moreover $h^0(N \otimes M) = 1$.

As N^{-1} is a subsheaf of M, the induced map $H^1(\operatorname{Hom}(M, E)) \to H^1(\operatorname{Hom}(N^{-1}, E))$ is surjective. Since the latter space is nonzero, the map $N^{-1} \to M$ does not lift to a generic extension $0 \to E \to W \to M \to 0$. In particular, a generic such W has a well-defined theta divisor. Moreover, it is easy to find an extension W satisfying $h^0(P \otimes W) > 0$ for at least one P with $h^0(P \otimes E) = 0$. For such a W the divisor Θ_W also has a component containing P, hence distinct from Θ_V, and so is reducible. □

We summarize our results as follows:

Corollary 3.4. *Let X be a curve of genus $g \geq 5$. Let Θ_V be the theta divisor of a generic stable bundle V of rank $n \geq 1$. Let t be a positive integer with $t < n(g-1)$. Then for any rank $tn + 1 + m \geq tn + 2$, there exists a stable bundle W of rank $tn+1+m$ such that Θ_W exists and contains $(t-1)\Theta_V$ as a subscheme. In particular, there exist stable bundles of all ranks ≥ 4 (resp., ≥ 5) with reducible (resp., reducible and nonreduced) theta divisors.*

Remark 3.5. The only place we use the fact that $g \neq 4$ is to ensure that we can choose t with $3 \leq t < n(g-1)$ when $n = 1$. If we let $g = 4$ and choose $n = 2$ and $t = 3$, then the construction gives stable bundles of rank 6 with nonreduced theta divisors.

Remark 3.6. Consider a generic W with extension class $\delta(W)$ and theta divisor $(t-1)\Theta_V + R_W$. Following [8, §5], we can give a geometric description of R_W.

Let N be a generic line bundle of degree $g - 1$ satisfying $h^0(N \otimes E) = 0$ and $h^0(N \otimes M) = 1$. Then by Riemann–Roch,

$$\operatorname{Im}\left(m_N \colon H^0(K_X N^{-1} \otimes E^*) \otimes H^0(N \otimes M) \to H^0(K_X \otimes E^* \otimes M)\right)$$

is of dimension 1. The association $N \mapsto \operatorname{Im} m_N$ defines a rational map

$$\mu \colon J \dashrightarrow \mathbb{P}H^0(K_X \otimes E^* \otimes M) \cong \mathbb{P}H^1(M^* \otimes E)^*.$$

Since m_N is injective, N belongs to the indeterminacy locus of μ if and only if $h^0(N \otimes E) > 0$ or $h^0(N \otimes M) > 1$.

We write R'_W for the complement of the indeterminacy locus in R_W, and H_W for the hyperplane defined by $\delta(W)$ on $\mathbb{P}H^0(K_X \otimes E^* \otimes M)$.

Lemma 3.7. *The set-theoretic intersection of H_W and $\mu(J)$ is exactly R'_W.*

Proof. Suppose $N \in J$ lies outside the indeterminacy locus of μ. We have

$$0 \to H^0(N \otimes W) \to H^0(N \otimes M) \xrightarrow{\cdot \cup \delta(W)} H^1(N \otimes E) \to \cdots$$

Then $N \in R_W$ if and only if the (up to scalar) unique section σ of $N \otimes M$ satisfies $\sigma \cup \delta(W) = 0$. By hypothesis, $h^0(N \otimes M) = 1 = h^1(N \otimes E)$, so this is equivalent to

$$\cdot \cup \delta(W) = 0 \ \in \ \mathrm{Hom}\left(H^0(N \otimes M), H^1(N \otimes E)\right).$$

Now it is well known that via Serre duality, the cup product map

$$H^1(\mathrm{Hom}(M, E)) \to \mathrm{Hom}\left(H^0(N \otimes M), H^1(N \otimes E)\right)$$

is dual to m_N. Thus $h^0(N \otimes W) > 0$ if and only if $m_N^* \delta(W) = 0$; in other words, $\mathrm{Im}\, m_N$ belongs to the kernel of the linear form on $H^0(K_X \otimes E^* \otimes M)$ defined by $\delta(W)$. The lemma follows. $\square$

4. Symplectic extensions and liftings

Recall that a vector bundle W is *symplectic* if there is an antisymmetric isomorphism $W \xrightarrow{\sim} W^*$; equivalently, if there exists a global bilinear nondegenerate antisymmetric form on W. In this section we gather some facts about such bundles.

Criterion 4.1. *Let $E \to X$ be a simple vector bundle and $0 \to E \to W \to E^* \to 0$ an extension of class $\delta(W) \in H^1(\mathrm{Hom}(E^*, E)) = H^1(E \otimes E)$. Then W carries a symplectic form with respect to which E is isotropic if and only if $\delta(W)$ belongs to the subspace $H^1(\mathrm{Sym}^2 E)$.*

Proof. This is a special case of [7, Criterion 2.1]. $\square$

Proposition 4.2. *Let E be any vector bundle and $F \subseteq E$ a subbundle. Then*

$$(F \otimes E) \cap \mathrm{Sym}^2 E = \mathrm{Sym}^2 F.$$

Proof. The question is local. For some $x \in X$, suppose

$$\sum e_i \otimes f_i \ \in \ (E \otimes F)|_x \cap \mathrm{Sym}^2 E|_x,$$

where each $e_i \in E|_x$ and $f_i \in F|_x$. Furthermore, we assume that the sum is of minimal length (equal to the rank of the associated map $E^*|_x \to F|_x$). Then

$$\sum e_i \otimes f_i = \sum f_i \otimes e_i$$

and so by minimality $e_i = f_{\rho(i)}$ for some permutation ρ of the indices. Hence all the e_i belong to $F|_x$. The proposition follows. $\square$

Now let E be a vector bundle and $F \subset E$ a subbundle, and write $G := E/F$. If $h^0(G \otimes E) = 0$, then using Proposition 4.2 we find a diagram

(4.1)
$$\begin{tikzcd}
 & & 0 \arrow[d] & H^0(\wedge^2 G) \arrow[d] & \\
0 \arrow[r] & H^1(\mathrm{Sym}^2 F) \arrow[r] \arrow[d] & H^1(\mathrm{Sym}^2 E) \arrow[r, "b"] \arrow[d] & H^1\left(\frac{\mathrm{Sym}^2 E}{\mathrm{Sym}^2 F}\right) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(F \otimes E) \arrow[r] \arrow[d] & H^1(E \otimes E) \arrow[r] \arrow[d] & H^1(G \otimes E) \arrow[r] \arrow[d] & 0 \\
H^0(\wedge^2 G) \arrow[r] & H^1\left(\frac{F \otimes E}{\mathrm{Sym}^2 F}\right) \arrow[r] \arrow[d] & H^1(\wedge^2 E) \arrow[r] \arrow[d] & H^1(\wedge^2 G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Lemma 4.3. *Let E, F and G be as above. Then the subbundle $G^* \subseteq E^*$ lifts to W if and only if $\delta(W)$ belongs to the preimage of $H^0(\wedge^2 G)$ via the map b in (4.1). In particular, if $h^0(\wedge^2 G) = 0$ then G^* lifts to W if and only if $\delta(W) \in H^1(\mathrm{Sym}^2 F)$.*

Proof. It is well known that G^* lifts to W if and only if $\delta(W)$ belongs to
$$\mathrm{Ker}\left(H^1(E \otimes E) \to H^1(G \otimes E)\right) = H^1(F \otimes E).$$
Thus we must describe $H^1(F \otimes E) \cap H^1(\mathrm{Sym}^2 E)$. In (4.1), we have
$$H^1(F \otimes E) \cap H^1(\mathrm{Sym}^2 E) = \mathrm{Ker}\left(H^1(\mathrm{Sym}^2 E) \to H^1(E \otimes E) \to H^1(G \otimes E)\right).$$
By commutativity, this coincides with
$$\mathrm{Ker}\left(H^1(\mathrm{Sym}^2 E) \xrightarrow{b} H^1\left(\frac{\mathrm{Sym}^2 E}{\mathrm{Sym}^2 F}\right) \to H^1(G \otimes E)\right).$$
Thus $H^1(F \otimes E) \cap H^1(\mathrm{Sym}^2 E)$ is the preimage of $H^0(\wedge^2 G)$ by b. The lemma follows. □

5. Symplectic bundles with reducible and nonreduced theta divisors

Here we adapt the construction of §3 to produce stable symplectic bundles with reducible and nonreduced theta divisors. As before, suppose X has genus $g \geq 5$. Let L be a line bundle of degree -1 and V a generic stable bundle of rank $n \geq 1$ and degree zero. Let E be a generic extension $0 \to L \to E \to V \otimes \mathbb{C}^t \to 0$, where $t < n(g-1)$.

Lemma 5.1. *A generic symplectic extension $0 \to E \to W \to E^* \to 0$ is a stable vector bundle.*

Proof. From Lemma 2.2 it follows that every nonzero quotient of E^* has positive degree, and hence every proper subbundle of E^* has nonnegative degree. Since W is

nonsplit, it is semistable. Suppose $F \subset W$ is a subbundle of degree 0. Then we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] & W \arrow[r] & E^* \arrow[r] & 0 \\ 0 \arrow[r] & F_1 \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{5.1}$$

where F_1 is a subbundle of E and F_2 a subsheaf of E^*. There are a priori three possibilities:

(i) If $F_1 \neq 0$ then $\deg F_1 \leq -1$ by Lemma 2.2, whence $F_2 = E^*$ and $\deg F_1 = -1$. By Corollary 2.5 (1), the bundle F_1 is an extension $0 \to L \to F_1 \to V \otimes \Lambda \to 0$ where Λ is a (possibly zero-dimensional) subspace of $\mathbb{C}^t$.

(ii) If $F_1 = 0$ and $\operatorname{rk} F_2 < \operatorname{rk} E$ then $F = F_2$ is a proper subbundle of degree zero. By dualizing the statement of Lemma 2.3, we see that F is of the form $V^* \otimes \Pi$ for a nonzero subspace $\Pi \subset (\mathbb{C}^t)^*$.

(iii) If $F_1 = 0$ and $\operatorname{rk} F_2 = \operatorname{rk} E$ then $F = F_2$ is an elementary transformation of E^* along a torsion sheaf of length 1.

We deal with each of these possibilities in turn:

(i) Here we obtain a diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & E^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & E^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \otimes (\mathbb{C}^t/\Lambda) \arrow[r, "\sim"] \arrow[d] & V \otimes (\mathbb{C}^t/\Lambda) \arrow[r] \arrow[d] & 0 & \\ & 0 & 0 & & \end{tikzcd}$$

Dualizing this diagram, we see that W^* contains a subbundle of the form $V^* \otimes (\mathbb{C}^t/\Lambda)^*$ lifting from E^*. But since W is self-dual, this means that we are also in situation (ii). Thus it suffices to exclude possibility (ii) in general.

(ii) Clearly it suffices to treat the case $\dim \Pi = 1$. For any inclusion $V^* \otimes \Pi \to E^*$, it suffices to show that

$$\dim \operatorname{Ker} \left(j^* \colon H^1(\operatorname{Sym}^2 E) \to H^1(\operatorname{Hom}(V^* \otimes \Pi, E)) \right) + \dim \mathbb{P}^{t-1} < h^1(\operatorname{Sym}^2 E).$$

We write F_Π for the subbundle of E defined by the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & E \arrow[r] & V \otimes \mathbb{C}^t \arrow[r] & 0 \\ 0 \arrow[r] & L \arrow[r] \arrow[u] & F_\Pi \arrow[r] \arrow[u] & V \otimes \Pi^\perp \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

On the vector bundle level, clearly we have

$$\mathrm{Ker}\,(j^* \colon \mathrm{Hom}(E^*, E) \to \mathrm{Hom}(V^* \otimes \Pi, E)) \cong \mathrm{Hom}(F_\Pi^*, E).$$

Hence by Proposition 4.2, we have $\mathrm{Ker}\, j^* \cap \mathrm{Sym}^2 E = \mathrm{Sym}^2 F_\Pi$. Thus to show that situation (ii) does not arise in general, it suffices to show that

$$h^1(\mathrm{Sym}^2 F_\Pi) + h^0(\wedge^2(V \otimes \Pi)) + t - 1 < h^1(\mathrm{Sym}^2 E).$$

Since V is generic, $h^0(\wedge^2(V \otimes \mathbb{C}^t)) = 0$. Then the required inequality follows from a computation using Riemann–Roch and the fact that $h^0(\mathrm{Sym}^2 F_\Pi) = 0$.

(iii) It suffices to show that $\dim \psi\,(\Delta_{E\otimes E}) < h^1(\mathrm{Sym}^2 E) - 1$. This follows from

$$2tn + 1 < (tn+2) + \frac{(tn+1)(tn+2)}{2}(g-1) - 1,$$

which is a consequence of the inequalities $g - 1 \geq 4$ and $\frac{(tn+1)(tn+2)}{2} \geq tn$.

In summary, a general symplectic extension $0 \to E \to W \to E^* \to 0$ admits no subbundles of nonnegative degree. □

We now describe the theta divisor of a generic such W.

Theorem 5.2. *A general symplectic extension $0 \to E \to W \to E^* \to 0$ has a reducible theta divisor Θ_W which scheme-theoretically contains $(t-1)\,(\Theta_V + \Theta_{V^*})$. In particular, if $t \geq 3$ then Θ_W is nonreduced.*

Proof. Exactly as in the proof of Theorem 3.3, we see that the theta divisor of a generic such W, if it exists, scheme-theoretically contains $(t-1)\Theta_V$. By Serre duality (see also Beauville [4, §2]) we have $\iota^*\Theta_W = \Theta_W$, where ι is the involution of J induced by $N \mapsto K_X N^{-1}$. In general, $\Theta_{V^*} = \iota^*\Theta_V$ via Serre duality. Since V is generic of degree zero, we may assume that $\Theta_V \neq \Theta_{V^*}$. Hence Θ_W scheme-theoretically contains $(t-1)\,(\Theta_V + \Theta_{V^*})$.

It remains to check that Θ_W is defined. Choose a generic line bundle $N \in J \backslash \mathrm{Supp}\,(\Theta_V + \Theta_{V^*})$. By Riemann–Roch and genericity, $h^0(N \otimes E^*) = 1$, and the corresponding map $j \colon N^{-1} \to E^*$ is a vector bundle injection. Dualizing, we obtain an exact sequence $0 \to G \to E \to N \to 0$, where $G := (E^*/N^{-1})^*$. Now we claim:

$$h^1(\mathrm{Sym}^2 E) - h^1\left(\mathrm{Sym}^2 G\right) > 0. \tag{5.2}$$

Since $\mathrm{Sym}^2 G$ is a subbundle of $\mathrm{Sym}^2 E$, it has no global sections. Then a computation with Riemann–Roch shows that the left hand side of (5.2) is equal to 1 (the expected value).

By Lemma 4.3, we have $h^0(N \otimes W) = 0$ for a generic symplectic extension $0 \to E \to W \to E^* \to 0$. Hence such a W has a well-defined theta divisor Θ_W. As in Theorem 3.3, we check that Θ_W in general has at least one component distinct from Θ_V. This completes the proof of Theorem 5.2. □

As before, we summarize as follows:

Corollary 5.3. *Let Θ_V be the theta divisor of a generic stable bundle V of rank $n \geq 1$. Let t be a positive integer with $t < n(g-1)$. Then there exist stable symplectic vector bundles W of rank $2(tn+1)$ such that Θ_W exists and contains $(t-1)\,(\Theta_V + \Theta_{V^*})$ as a subscheme. In particular, for all even ranks $r \geq 6$ (resp., $r \geq 8$) there exist*

stable symplectic vector bundles with reducible (resp., reducible and nonreduced) theta divisor.

Remark 5.4. As noted previously in Remark 3.5, the only place where $g \geq 5$ is used is to ensure that when $n = 1$ we can choose t with $3 \leq t < g - 1$. If $g = 4$ then we can take $n = 2$ and $t = 3$, and obtain a symplectic bundle of rank 8 with nonreduced theta divisor.

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